Construction of Coefficient Inequality for a New Subclass of Class of Starlike Analytic Functions

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Abstract
In this paper, we will discuss a newly constructed subclass of analytic starlike functions by which we will be obtaining sharp upper bounds of the functional \( |a_3 - \mu a_2^2 | \) for the analytic function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n , |z| < 1 \) belonging to this subclasses.

Keywords: Univalent functions; Starlike functions; Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

1. Introduction: Let \( \mathcal{A} \) denote the class of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]  \hspace{1cm} (1.1)

which are analytic in the unit disc \( \mathbb{E} = \{ z : |z| < 1 \} \). Let \( \mathcal{S} \) be the class of functions of the form (1.1), which are analytic univalent in \( \mathbb{E} \).

In 1916, Bieberbach \([7],[8]\) proved that \( |a_2| \leq 2 \) for the functions \( f(z) \in \mathcal{S} \). In 1923, Löwner [5] proved that \( |a_3| \leq 3 \) for the functions \( f(z) \in \mathcal{S} \).

With the known estimates \( |a_2| \leq 2 \) and \( |a_3| \leq 3 \), it was natural to seek some relation between \( a_3 \) and \( a_2^2 \) for the class \( \mathcal{S} \). Fekete and Szegő [9] used Löwner’s method to prove the following well known result for the class \( \mathcal{S} \).

Let \( f(z) \in \mathcal{S} \), then

\[ |a_2 - \mu a_2^2 | \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0; \\ 1 + 2 \exp\left(\frac{2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} \]  \hspace{1cm} (1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes \( \mathcal{S} \) (See Chhichra[1], Babalola[6]).

Let us define some subclasses of \( \mathcal{S} \).

We denote by \( S^n \), the class of univalent starlike functions \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} \) and satisfying the condition
We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_{n}z^{n}, z \in A$ and satisfying the condition
\[ \text{Re} \left( \frac{zh'(z)}{h'(z)} \right) > 0, z \in \mathbb{E}. \] (1.3)

A function $f(z) \in A$ is said to be close to convex if there exists $g(z) \in S^{*}$ such that
\[ \text{Re} \left( \frac{zf'(z)}{g'(z)} \right) > 0, z \in \mathbb{E}. \] (1.4)

The class of close to convex functions is denoted by $C$ and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

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It is obvious that $S^{*}(A,B)$ is a subclass of $S^{*}$ and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as $F(z) \in \mathcal{A}$, and we will denote this class as $f(z) \in \Sigma S^{*}(\alpha)$.

Symbol $<$ stands for subordination, which we define as follows:

**Principle of Subordination:** Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to $F(z)$ in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$, $z \in \mathbb{E}$ and we write $f(z) < F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_{n}z^{n}, w(0) = 0, |w(z)| < 1$. It is known that $|d_{1}| \leq 1, |d_{2}| \leq 1 - |d_{1}|^{2}$.

**2. PRELIMINARY LEMMAS:** For $0 < c < 1$, we write $w(z) = \left( \frac{z+c}{1+cz} \right)$ so that

\[ \frac{1+cw(z)}{1+zw(z)} = 1 + (A-B)c_{1}z + (A-B)(c_{2}-Bc_{1}^{2})z^{2} + \cdots \] (2.1)

**3. MAIN RESULTS**

**THEOREM 3.1:** Let $f(z) \in f(z) \in \Sigma S^{*}(\alpha)$, then

\[ |\alpha_{3} - \mu \alpha_{2}^{2}| = \left\{ \begin{array}{cl}
\frac{2\alpha}{(\alpha+1)^{3}} [5\alpha^{2} + 10\alpha - 3 - 8\alpha \alpha(\alpha + 1)] ; & \text{if } \mu \leq \frac{4\alpha^{2} + 8\alpha - 4}{8\alpha(\alpha + 1)} (3.1) \\
\frac{2\alpha}{(\alpha + 1)^{3}} [8\mu(\alpha + 1) - 65 - 10\alpha + 3] ; & \text{if } \mu \geq \frac{6\alpha^{2} + 12\alpha - 2}{8\alpha(\alpha + 1)} (3.3)
\end{array} \right. \]

The results are sharp.

**Proof:** By definition of $f(z) \in f(z) \in \Sigma S^{*}(\alpha)$, we have
Expanding the series (3.4), we get
\[
1 + \alpha_2 \left( \frac{\alpha + 1}{2\alpha} \right) + \frac{z}{2} \left( (2\alpha_3 - \alpha_2^2) \left( \frac{\alpha + 1}{\alpha} \right) + \left( \frac{1-\alpha}{2\alpha_2} \right) \alpha_2 \right) + \ldots = (1 + 2c_1z + 2(c_1^2 + c_2)z^2 + z^3(2c_3 + 4c_1c_2 + c_1^3) + \ldots).
\] (3.5)

Identifying terms in (3.5), we get
\[
\alpha_2 = \frac{4c_1}{\alpha + 1}
\] (3.6)
\[
\alpha_3 = \left( \frac{2\alpha}{\alpha + 1} \right) c^2 + c_2 + \frac{4c_1^2}{(\alpha + 1)^2} [\alpha^2 + 2\alpha - 1]
\] (3.7)

From (3.6) and (3.7), we obtain
\[
\alpha_3 - \mu \alpha_2^2 = c^2 \left( \frac{2\alpha}{\alpha + 1} + \frac{8\alpha(\alpha^2 + 2\alpha - 1)}{(\alpha + 1)^3} - \frac{16\mu \alpha^2}{(\alpha + 1)^2} \right) + c_2 \left( \frac{2\alpha}{\alpha + 1} \right)
\] (3.8)

Taking absolute value, (3.8) can be rewritten as
\[
|\alpha_3 - \mu \alpha_2^2| \leq \left| \frac{2\alpha}{\alpha + 1} + \frac{8\alpha(\alpha^2 + 2\alpha - 1)}{(\alpha + 1)^3} - \frac{16\mu \alpha^2}{(\alpha + 1)^2} \right| |c_1|^2 + |c_2| \left| \frac{2\alpha}{\alpha + 1} \right|
\] (3.9)

Using (1.11) in (3.9), we get
\[
|\alpha_3 - \mu \alpha_2^2| \leq \left( \frac{2\alpha}{(\alpha + 1)^3} \right) \left[ (5\alpha^2 + 10\alpha - 3) |8\mu \alpha (\alpha + 1)| - \frac{2\alpha}{\alpha + 1} \right] |c_1|^2 + \frac{2\alpha}{\alpha + 1}
\] (3.10)

Case I: \( \mu \geq \frac{5\alpha^2 + 10\alpha - 3}{8\alpha(\alpha + 1)} \). (3.10) can be rewritten as
\[
|\alpha_3 - \mu \alpha_2^2| \leq \frac{2\alpha}{(\alpha + 1)^3} \left[ 8\mu \alpha (\alpha + 1) - (6\alpha^2 + 12\alpha - 2) \right] |c_1|^2 + \frac{2\alpha}{\alpha + 1}
\] (3.11)

Subcase I (a): \( \mu \geq \frac{6\alpha^2 + 12\alpha - 2}{8\alpha(\alpha + 1)} \). Using (1.11), (3.11) becomes
\[
|\alpha_3 - \mu \alpha_2^2| \leq \frac{2\alpha}{(\alpha + 1)^3} \left[ 8\mu \alpha (\alpha + 1) - 5\alpha^2 - 10\alpha + 3 \right].
\] (3.12)

Subcase I (b): \( \mu < \frac{6\alpha^2 + 12\alpha - 2}{8\alpha(\alpha + 1)} \). We obtain from (3.11)
\[
|\alpha_3 - \mu \alpha_2^2| \leq \frac{2\alpha}{\alpha + 1}
\] (3.13)

Case II: \( \mu < \frac{5\alpha^2 + 10\alpha - 3}{8\alpha(\alpha + 1)} \)

Preceding as in case I, we get
\[
|\alpha_3 - \mu \alpha_2^2| \leq \frac{2\alpha}{\alpha + 1} + \frac{2\alpha}{(\alpha + 1)^3} \left[ 4\alpha^3 + 8\alpha - 4 - 8\mu \alpha (\alpha + 1) \right] |c_1|^2.
\] (3.14)
Subcase II (a): $\mu \leq \frac{4\alpha^2 + 8\alpha - 4}{8\alpha(\alpha + 1)}$

(3.14) takes the form

$|a_3 - \mu a_2^\alpha| \leq \frac{2\alpha}{(\alpha + 1)^3} \left[ 5\alpha^2 + 10\alpha - 3 - 8\mu\alpha(\alpha + 1) \right]$  \hfill (3.16)

Subcase II (b): $\mu > \frac{4\alpha^2 + 8\alpha - 4}{8\alpha(\alpha + 1)}$

Preceding as in subcase I (a), we get

$|a_3 - \mu a_2^\alpha| \leq \frac{2\alpha}{\alpha + 1}$  \hfill (3.17)

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = (1 + az)^b$$

Where $a = \frac{((2\alpha + \beta - 3\beta)^2(1 - \alpha)\beta(\beta - 3) + 4\alpha(1 - \beta)(\beta + 2))a_2^2 - 4(3\alpha + \beta - 4\alpha \beta)a_3}{(2\alpha + \beta - 3\beta)a_3}$

And $b = \frac{(2\alpha + \beta - 3\beta)^2a_1^2}{((2\alpha + \beta - 3\beta)^2(1 - \alpha)\beta(\beta - 3) + 4\alpha(1 - \beta)(\beta + 2))a_2^2 - 4(3\alpha + \beta - 4\alpha \beta)a_3}$

Extremal function for (3.2) is defined by $f_2(z) = z(1 + Bz^2)^{\frac{A - B}{2B}}$.

Corollary 3.2: Putting $\alpha = 1, \beta = 0$ in the theorem, we get

$|a_3 - \mu a_2^\alpha| \leq \begin{cases} 1 - \mu, & \text{if } \mu \leq 1; \\ \frac{1}{3}, & \text{if } 1 \leq \mu \leq \frac{4}{3}; \\ \mu - 1, & \text{if } \mu \geq \frac{4}{3} \end{cases}$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

Corollary 3.3: Putting $A = 1, B = -1$ and $\alpha = 0, \beta = 1$ in the theorem, we get

$|a_3 - \mu a_2^\alpha| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq \frac{1}{2}; \\ \frac{1}{2}, & \text{if } \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, & \text{if } \mu \geq 1 \end{cases}$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

References: